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ADDRESS

Delivered by the Chairman, Professor J. C. Adams, on Presenting the Gold Medal of the Society to M. Charles Delaunay.

Gentlemen,—It has been announced to you that the Society's Medal has been awarded to M. Ch. Delaunay for his great work on the Theory of the Moon.

The illness of our excellent President having made it impossible for him to be present on this occasion, the Council have done me the honour to request that I would occupy the chair, and in his stead lay before you the grounds of their award. I have acceded to their wishes with the more readiness because I have given some attention to special branches of the Lunar Theory, and my study of M. Delaunay's work has led me to form the highest opinion of its merits.

Of all the problems presented to us by physical astronomy none has so much engaged the attention of mathematicians as that of the determination of the motion of our satellite. The theoretical interest as well as the great practical importance of the results, has proved an irresistible attraction, and the mathematical difficulties have merely acted as a stimulus to the invention of various methods of surmounting them. It is fortunate that this has been the case, as the excessive labour involved in any theory of the Moon approaching to completeness, might otherwise have proved too great for human perseverance. The foundations of the theory were laid by Newton in his *Principia;* and although his investigations are only fragmentary, being simply intended to show

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how some of the leading lunar inequalities may be deduced from theory, yet they form one of the most admirable portions of that immortal work. Towards the middle of the eighteenth century the theory was more systematically entered upon by Clairaut, D'Alembert, and Euler, who severally showed that the theory was competent to give very approximate values of all the inequalities which were then recognised by observation.

Still the theory was far from being sufficiently perfect to serve as a foundation for lunar tables accurate enough for the uses of navigation. This degree of accuracy was first attained by the tables of Mayer, who not only carried the approximations to the values of the coefficients of the various lunar inequalities further than his predecessors had done, but also corrected the theoretical coefficients thus obtained by comparison with his own observations. The theory was greatly advanced by Laplace, not only by his more accurate theoretical determination of the coefficients, but also by several important discoveries, especially that of the cause of the Moon's secular acceleration.

The improvements in the lunar tables, however, which were made successively by Bürg and Burckhardt, were founded, not on theory, but on comparison of the former tables with observations; and the empirical tables thus produced were far more accurate than any that could have been formed at that time by theory alone. Dissatisfied with this state of things, and wishing to see astronomy founded exclusively on the law of attraction, only borrowing from observation the necessary data, Laplace induced the Academy of Sciences to propose for the subject of the mathematical prize which it was to award in 1820 the formation, by theory alone, of lunar tables as exact as those which had been constructed by theory and observation combined. The prize was divided between two memoirs — one by M. Damoiseau, the other being the joint production of MM. Plana and Carlini. Damoiseau's memoir is printed in the third volume of the *Recueil des Savants Etrangers*. Plana's great work on the lunar theory, which appeared in 1832, is the development of the joint memoir by himself and Carlini. By these important works an immense advance was made in the theory, the approximations being carried to such an extent that the resulting coefficients were comparable in accuracy with those given by observation. In 1824 Damoiseau published tables founded entirely on his theory, which were found to be quite as exact as those of Burckhardt.

Both Damoiseau and Plana, following the example of Laplace, start from differential equations in which the Moon's longitude is taken as the independent variable; and after the equations have been integrated, they obtain the values of the Moon's coordinates in terms of the time by reversion of series. An important innovation, however, was introduced by Plana in the mode of conducting the investigation and exhibiting the results. The values of the Moon's co-ordinates being developed in series of sines and cosines of angles which vary uniformly with

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the time, the coefficients of the several terms of these series will depend on the eccentricities of the orbits of the Sun and Moon, the inclination of the Moon's orbit to the plane of the ecliptic, the ratio of the mean motions of the Sun and Moon, and the ratio of their mean distances from the Earth. Now Damoiseau, in common with all previous writers, having assumed certain values of the quantities just mentioned as given by observation, contented himself with determining the numerical values of the coefficients. Although this is all that is required for the construction of tables, yet, from a theoretical point of view, it leaves the mind unsatisfied, inasmuch as any coefficient in its numerical form shows no trace of its composition, that is of the manner in which its value depends on the value of the assumed elements. The several coefficients are far too complicated functions of the elements to be represented analytically, except in the form of infinite series, and Plana, accordingly, developes these coefficients in such series, proceeding by powers and products of the eccentricities, the tangent of the inclination, the ratio of the Sun's mean motion to that of the Moon, and the ratio of the Moon's mean distance to that of the Sun, all these quantities being assumed to be small, and the last mentioned ratio, which is much smaller than the others, being considered as a quantity of the second order.

In this mode of development, the numerical factor which enters into any term of the coefficient of any of the lunar inequalities is an ordinary fraction which admits of being determined not merely approximately, but with absolute accuracy. It is easy to see what great facilities are afforded by this circumstance for the verification of the work by a comparison of the results obtained by different methods. The greater or less degree of approximation will thus depend on the greater or less number of terms taken into account in the several series.

The numerical values of the several elements are not substituted in the formulæ until the work is completed, and this is attended with the important advantage that when a comparison of the theory with observation has supplied more accurate values of the elements, their corrected values can be at once substituted in the same formulæ, without requiring any additional work.

On the other hand, if the numerical values of the elements be introduced into the calculations from the first, then if it is desired to introduce corrected values of the elements, much additional investigation will be required for the purpose.

No doubt the labour required in order to obtain a given amount of numerical accuracy by this method is very much greater than is required when each coefficient, instead of consisting of a series of terms, is reduced to a simple numerical quantity, but the great theoretical advantage of knowing the composition of every coefficient in terms of the elements well repays the additional labour.

The degree of convergence of the series obtained for the

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several coefficients is in general sufficiently rapid, but in some few of the coefficients, on the contrary, the convergence is so slow, at least in the leading terms, that it is necessary to take into account terms which are analytically of a higher order than those to which the approximation is in general limited.

Thus Plana, who proposed to himself to determine the lunar inequalities completely to the fifth order, found it necessary in special cases to carry the approximation to the seventh and even to the eighth order, and in several cases he also added an estimated value of the remainder of the series founded on the observed law of diminution of the calculated terms.

Soon after the publication of Plana's great work, Sir John Lubbock formed the plan, which he partly carried out in his various tracts on the theory of the Moon, of verifying Plana's results by a totally different method, starting from differential equations in which the time is taken as the independent variable, and thus avoiding the necessity of reversion of series.

Later, M. de Pontécoulant undertook the same work on a similar plan, and carried it out more completely in the fourth volume of his *Théorie Analytique de Système du Monde.*

These works, while they corrected some errors which had crept into Plana's computations, confirmed their wonderful general accuracy, and with some few exceptions they do not extend the approximation beyond the order to which Plana restricts himself.

Meantime, M. Hansen had undertaken a completely new investigation of the lunar theory, by a remarkable method peculiar to himself and explained in his *Fundamenta nova investigationis orbitæ veræ quam Luna perlustrat*, which appeared in 1838.

In applying the method described in this work to the case of the Moon, M. Hansen throughout employs numerical values of the elements of the Moon's orbit, and consequently the coefficients of the lunar inequalities as obtained by him are also purely numerical. The process is one of successive approximations, which are repeated again and again until the values of the inequalities which are found from the last approximation sensibly coincide with those which were assumed in entering upon that approximation.

The numerical values of the coefficients thus finally obtained are undoubtedly very exact. The slight corrections which these coefficients still require are probably chiefly due to the small corrections required by the numerical elements on which the calculations are based, and in the method employed no provision is made for taking into account the effect of these corrections.

From his formulæ, M. Hansen constructed tables of the Moon, which were published in 1857, at the expense of the British Government; and these tables, having been found far superior in accuracy to all others, are now exclusively employed in the calculation of ephemerides.

A detailed account of the calculations leading to M. Hansen's

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last approximation, was given by him in the two parts of his *Darlegung der Theoretischen Berechnung der in den Mondtafeln angewandten Störungen*, which severally appeared in 1862 and 1864.

After the great works, to which we have thus briefly referred, had been either completed or were in progress, it might have been supposed that the matter was exhausted.

Our Associate M. Delaunay, however, was not of this opinion. Having devised, so long ago as 1846, a perfectly original and singularly beautiful method of integrating the differential equations of the Moon's motion, he determined to apply this method to the complete re-investigation of the theory, and to carry on the approximation to a much greater extent than had been done by his predecessors. The principal fruits of his labours, to which he has devoted himself with almost unexampled perseverance for so many years, are contained in the magnificent volumes which the Imperial Academy of Sciences have done both M. Delaunay and themselves the honour of publishing among the volumes of their *Memoirs*. It is for this great work that your Council have awarded to M. Delaunay the Society's medal.

Strongly impressed with the advantages of determining the co-efficients of the lunar inequalities in the analytical form, both as affording a solution more complete in itself and more satisfactory to the mind, as well as one offering facilities for the comparison of the results of different investigations, M. Delaunay did not hesitate to follow the example set in this respect by M. Plana, notwithstanding the immense length of the necessary calculations. M. Delaunay's results are thus obtained in a form which makes them directly comparable with those of M. Plana, while the methods employed in obtaining them are wholly different.

M. Delaunay chooses the *time* as the independent variable, and takes as his starting-point the differential equations furnished by the theory of the variation of the arbitrary constants. In an able Memoir which appeared in 1833, Poisson had advocated the employment of these equations in the theory of the Moon's motion, and he applied them to the discussion of some special points of that theory. These equations had been long used, almost exclusively, for the determination of the perturbations of the planets, and they offer peculiar advantages in the treatment of the secular inequalities and those of long period. In the case of the Moon, however, in consequence of the large perturbations caused by the disturbing force of the Sun, the ordinary mode of integrating these equations by successive approximations soon leads into calculations of inextricable complexity. In fact, these equations give the differential coefficients of the several elliptic elements taken with respect to the time, in terms of the elements themselves. In the case of the planets, where the disturbing forces are so small compared with the predominant central force of the Sun, very approximate values of the disturbed elements may be found by substituting in the values of the differential coefficients,

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the undisturbed instead of the disturbed values of the elements, and then integrating.

The perturbations of the elements thus found are said to be due to the first power of the disturbing force. If now the approximate values of the disturbed elements be substituted in the differential equations, and these be again integrated, we shall obtain a second approximation to the values of the disturbed elements, and the additional terms thus found are said to depend on the square of the disturbing force. In the theories of the planets it is only in special cases that terms depending on the square of the disturbing force need be taken into account, and it is scarcely ever necessary to consider terms of the next order of approximation.

In the case of the Moon, however, it would be necessary to repeat the process of approximation at least four or five times, in order to obtain results of the accuracy required in the present state of the theory. If we consider that the disturbing function consists of a great number of terms, and that each term gives rise to a corresponding term in the value of each of the disturbed elements, while powers and products of the corrections of all the elements in every possible combination, up to a certain order, have to be taken into account, it may be readily imagined how impracticable it would be by such a process to carry on the approximation to a greater extent than has been already done by Plana. Every process in which the approximations require to be repeated several times, is subject to the inconveniences that have been described, and these inconveniences are much greater when, as in the present case, we have to make successive approximations to the values of the *six* elements of the orbit, instead of to the values of the *three* co-ordinates of the Moon.

It was with the view of avoiding this excessive complication of the method of successive approximations that M. Delaunay devised his method of integrating the differential equations of the Moon's motion. The fundamental idea of this method consists in attacking the difficulty by small portions at a time, and in replacing these extremely complicated successive approximations by a much greater number of distinct operations, each of which is comparatively simple, so that it may be carried out to any degree of exactness that may be desirable, while the mind is relieved by being able readily to embrace the whole of each operation in one view.

It is difficult, without the use of algebraical symbols to give an idea of M. Delaunay's beautiful method, but I must endeavour, in some measure, to fulfil this task, and I must crave your indulgence should I fail in the attempt.

The theory of the variation of the arbitrary constants gives, as is well known, the differential co-efficients of the elliptic elements with respect to the time, in terms of the elements themselves and the partial differential co-efficients of a certain function, called the Disturbing Function, taken with respect to those ele-

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ments. By a proper choice of elements, the differential equations may be reduced to their simplest, or to what is called their *canonical* form. In this form the six elements are divided into three pairs, the elements of each pair being conjugate to each other. Then the differential coefficient of any element with respect to the time is simply equal to the partial differential coefficient of the disturbing function taken with respect to the element which is conjugate to the former, the partial differential coefficients which occur in the two equations corresponding to a pair of conjugate elements being affected with opposite signs.

The disturbing function may be readily developed in a series of periodic terms involving cosines of angles, each of which is formed by the combination of multiples of the Moon's mean longitude, the distance of the Moon's perigee from its node, and the longitude of the node, together with angles which depend on the position of the disturbing bodies. The disturbing function likewise contains a non-periodic term, which, as well as the coefficients of the periodic terms, are all functions of the major semi-axis, the eccentricity and the inclination of the Moon's orbit.

Since the mean longitude of the Moon involves the time multiplied by the mean motion which is a function of one of the elements, it is obvious that the differentiation with respect to this element will give rise to terms in which the time occurs without its being included under a sine or a cosine. Such terms would render the equations very inconvenient for the determination of the lunar inequalities; and M. Delaunay accordingly avoids the introduction of them by taking the mean longitude itself instead of the *epoch* of mean longitude, as one of his elements, while by the simple yet novel expedient of adding to the disturbing function a non-periodic term which is a function of the major semi-axis alone and is independent of the disturbing forces, he preserves to the differential equations the same very simple form which they had at first. After this modification of the disturbing function, the time no longer enters into it explicitly except in so far as it is introduced by the values of the co-ordinates of the disturbing bodies, and consequently the difficulty which was before met with completely disappears.

The six elements employed by M. Delaunay are thus,—the Moon's mean longitude, the distance of the perigee of its orbit from the node, and the longitude of the node, which for distinction may be called the three *angular* elements, and three other elements which are respectively conjugate to the former, and which are determinate functions of the major semi-axis, the eccentricity and the inclination of the orbit.

The three co-ordinates of the Moon at any time are given in terms of the three angular elements and of the quantities last mentioned.

Now let us imagine, for a moment, that the disturbing function contained no periodic terms, but was reduced simply to its non-periodic part. Consequently the partial differential co-

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efficients taken with respect to the angular elements would all vanish, and therefore the three conjugate elements would be all constant, as well as the major semi-axis, the eccentricity and inclination, of which those elements are functions. Hence, again, the partial differential co-efficients taken with respect to the conjugate elements would be functions of those elements, and would therefore be constant. Hence each of the angular elements would consist of an arbitrary constant and a term proportional to the time, the multiplier of the time in each case being a known function of the three constant elements.

The object of M. Delaunay's method is, by means of a series of changes of the variables, to cause all the more important periodic terms to disappear from the disturbing function, one by one, while the differential equations continue to retain their canonical form, so that after each transformation we approach more nearly to the conditions of the ideal case which has just been considered.

In order to effect any one of these transformations, M. Delaunay supposes, for the moment, that the disturbing function is reduced to its non-periodic part, together with one of the periodic terms selected from among those which have the greatest influence in producing the lunar inequalities. With this simplified form of the disturbing function, the equations admit of being easily integrated. The elements with which we start may thus be expressed in terms of three new angular elements which vary uniformly with the time, and three new constant elements. M. Delaunay shows how the constant elements may be so chosen that they may be considered as respectively conjugate to the three new angular elements, so that, in fact, the quantities which are multiplied by the time in the expressions of these angular elements are respectively equal to the partial differential co-efficients of a function of the new constant elements taken with respect to these elements.

Having thus found the relations between the old set of elements and the new ones by means of the simplified form of the disturbing function, M. Delaunay now restores the complete value of that function, and chooses new elements which are connected with the old ones by exactly the same relations as in the case just considered. Of course the three new angular elements will no longer vary uniformly with the time, and the three elements respectively conjugate to these will no longer be constant.

When, by means of the proper formulæ of transformation, the new variables have been substituted for the old ones in the disturbing function and in the expressions of the Moon's co-ordinates, M. Delaunay shows that—

1st. One of the important terms of the disturbing function disappears, viz., the periodic term which was selected in the preliminary investigation.

2nd. Various inequalities corresponding to this term are introduced into the values of the three co-ordinates of the Moon.

3rd. The values of the six new variables in terms of the time

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are determined by differential equations of exactly the same form as those which determined the values of the six variables for which they have been substituted.

One of the periodic terms having been in this manner caused to disappear from the disturbing function, a new operation of exactly the same kind causes another term of this function to disappear; similarly a third term may be taken away by means of a third operation, and so on to any number of terms.

In this way, after a suitable number of operations of this kind have been effected, the disturbing function will have been simplified by the removal from it of its most important periodic terms, after which the further process of integration becomes simple enough to be treated in the same manner as if we were concerned with the perturbations of a planet or of the Sun.

The whole difficulty in the determination of the lunar inequalities is caused by the great magnitude of the disturbing force of the Sun. M. Delaunay has therefore at first confined his attention to the investigation of the irregularities which are produced by this disturbing force, and the two magnificent volumes before us are entirely occupied with this investigation. Thus he has provisionally left out of consideration the very small inequalities due to some secondary causes, such as the attraction of the planets and the figure of the Earth; and, besides, he has omitted to consider the perturbations of the Sun's apparent motion about the Earth, intending in a supplementary volume to take into account the effects due to these several causes.

By means of repeated applications of the beautiful method of transformation which I have above attempted to describe, M. Delaunay proceeds to get rid of all the periodic terms of the disturbing function due to the Sun's disturbing force, which are capable of producing inequalities in the co-ordinates of the Moon of an order inferior to the fourth. For this purpose fifty-seven such operations are required to be performed. When these have been effected, the periodic terms which remain in the disturbing function are so small that their powers and products may be neglected, and consequently the differential equations which determine the six elements last introduced in terms of the time, may be integrated at once. Since the values of the Moon's coordinates are known in terms of the elements just mentioned and the time, we have only to substitute the values of the elements that have been found, in order to determine the Moon's co-ordinates in terms of the time.

The values of the elements, however, that would be found in this way are very complicated, and therefore the substitutions which would be required in order to find the Moon's co-ordinates would be excessively long. M. Delaunay, accordingly, prefers to get rid of the remaining periodic terms in the disturbing function, one by one, by means of transformations exactly similar to those which have been already effected. In order to carry on the approximation to the extent which he desires, M. Delaunay finds it necessary to perform no less than 448 of these secondary

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operations, but each such operation becomes very simple, since the squares of the coefficients of the periodic terms under consideration may be neglected.

Thus, at length, by means of 505 transformations, all the periodic terms of the disturbing function are removed, and the problem is reduced to the ideal case which was considered at the outset of our account of M. Delaunay's method.

After each transformation, by making the proper substitutions in the expressions for the Moon's co-ordinates, those co-ordinates are obtained in terms of the system of elements last introduced, so that finally the three co-ordinates are known in terms of the three final constants and angles which vary uniformly with the time.

It has been already mentioned that Plana, in his great work on the Lunar Theory, determined the analytical values of the coefficients of the lunar inequalities as far as terms of the fifth order inclusive, and that he only carried on the development to a greater extent in cases where the slowness of the convergence of the series appeared to him to render it necessary to take into account terms of higher orders than the fifth.

M. Delaunay has proposed to himself to carry on the approximation so as to include all terms of the seventh order, and in cases where the series converge slowly to take into account terms of the eighth, and even of the ninth order.

Those who have had any experience in calculations of this nature will readily understand how enormously the labour required has been increased by thus adding two orders more to those which Plana has considered. It is not merely that the terms of higher orders are far more numerous than those of the lower, but also that each of the terms of the former kind is much more difficult to calculate, since it arises from a much greater number of combinations of terms of the inferior orders.

This enormous labour, which has occupied M. Delaunay for nearly twenty years, has been performed by him without assistance from any one. Indeed, from the nature of the calculations which are required, it would not have been easy to obtain any effective assistance. In order to insure accuracy, M. Delaunay has omitted no means of verification, and he has performed all the calculations, without exception, at two separate times, with a sufficient interval between them to prevent any special risk of committing the same error twice in succession.

The volumes before us are perfect models of orderly arrangement. Notwithstanding the great length and complication of the calculations, the whole work is so disposed that any part of it may be specially examined with the utmost readiness by any one who may wish to test its accuracy.

Finally, the analytical expressions which have been obtained for the Moon's co-ordinates are converted into numbers, by substituting for the elements the most accurate numerical values which the comparison of theory with observation has made known.

Such is an imperfect sketch of M. Delaunay's labours on the

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Theory of the Moon contained in these two magnificent volumes, the former of which appeared in 1860, and the latter in 1867. As I have already stated, they do not include a complete theory of the Moon, but only that which is by far the most difficult and complicated part of that theory, viz., the investigation of the perturbations due to the direct action of the Sun supposing its apparent motion about the Earth to be purely elliptic. Of the investigations which are required to take into account the remaining very small causes of disturbance, and which are intended by M. Delaunay to be included in a supplementary volume, some of the most important have been already completed by him, particularly the calculation of the Secular Variation of the Moon's Mean Motion, and the investigation of the long inequalities due to the action of *Venus*.

I understand also that M. Delaunay is engaged in the construction of new Lunar Tables founded upon his theory.

Your Council, however, has decided that we ought not to await the appearance of M. Delaunay's supplementary researches before we mark emphatically our sense of the value of his labours.

The present work is complete in itself; in it the very difficult and complicated problem of determining the Moon's motion is attacked by a perfectly original method, and that one as powerful and beautiful as it is new. The work has been planned with admirable skill and has been carried out with matchless perseverance. The result is an enduring scientific monument of which our age may well be proud, and which we are happy to distinguish, on this occasion of our fiftieth anniversary, with the highest marks of our approval which it is in our power to bestow.

The Chairman, then delivering the Medal to M. Delaunay, addressed him in the following terms:—

M. Delaunay, il ne me reste plus maintenant qu'à vous présenter cette médaille au nom de la Société Royale Astronomique, qui désire par ce tribut vous exprimer la haute appréciation qu'elle a de vos travaux. Notre Président regrette vivement que l'état de sa santé l'empêche de remplir cette tâche agréable. Il m'a prié de le remplacer dans cette circonstance, et je le fais avec d'autant plus de plaisir que depuis bien long-temps j'ai la plus grande estime pour vos hauts talents, et que j'ai étudié vos belles recherches avec la plus grande admiration, aussi je suis heureux de vous exprimer que notre Société vous a suivi dans votre immense travail avec le plus vif intérêt; et quoique ce travail ne soit pas entièrement terminé, elle sent qu'elle ne peut tarder plus long temps à reconnaître la haute valeur de vos recherches. Nous sommes heureux de vous voir au milieu de nous à cette occasion, et nous faisons des vœux pour que votre santé et vos forces puissent durer de longues années afin d'enrichir la science de plus en plus du fruit de vos grands talents.